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Lower Bounds for the Spread of a Hermitian Matrix

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ABSTRACT

Some simple lower bounds for the spread of a Hermitian matrix are derived. These bounds are explicit functions of the entries of the matrix, one of which is sharper than a recent result due to E. R. Barnes and A. J. Hoffman. Comparisons are made with several known results. © Elsevier Science Inc., 1997

1. INTRODUCTION

Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues of an $n \times n$ matrix A. The spread of A is defined by

$$s(A) = \max_{i,j} |\lambda_i - \lambda_j|.$$

This quantity of Hermitian matrices has applications in combinatorial optimization problems [2]. Several authors have given bounds for the spread. If $A = (a_{ij}) \in \mathbb{C}^{n \times n}$ is Hermitian, Mirsky [4] shows that

$$s^2(A) \geqslant \max_{i \neq j} \left\{ (a_{ii} - a_{jj})^2 + 4|a_{ij}|^2 \right\}.$$

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Recently Barnes and Hoffman [1, Corollary] derived the following sharper result,

$$s^2(A) \geqslant \max_{i,j}\left\{(a_{ii}-a_{jj})^2 + 2\sum_{k\neq i}|a_{ik}|^2 + 2\sum_{k\neq j}|a_{jk}|^2\right\}. \tag{1}$$

We will improve this bound further. Denote by $|\alpha|$ the cardinality of a finite set α. Johnson, Kumar, and Wolkowicz [3, Theorem 2.2] prove for Hermitian A that

$$s(A) \geqslant \max_{\alpha,\beta}\frac{2}{\sqrt{|\alpha||\beta|}}\left|\sum_{\substack{i\in\alpha\\ j\in\beta}} a_{ij}\right|, \tag{2}$$

where $\varnothing \neq \alpha, \beta \subset \{1,2,\dots,n\}$ and $\alpha\cap\beta=\varnothing$. We will give a lower bound which is, in some cases, sharper than (2).

For arbitrary A the following upper bound due to Scott [5] can be easily proved by the Gerschgorin circle theorem (see also the proof of Theorem 3.1 in [1]):

$$s(A) \leqslant \max_{i,j}\left\{|a_{ii}-a_{jj}| + \sum_{k\neq i}|a_{ik}| + \sum_{k\neq j}|a_{jk}|\right\}. \tag{3}$$

All the bounds we present are easy to calculate, since they are simple functions of the entries of the matrix. In Section 2 we derive several new lower bounds. We make comparisons by some examples in Section 3.

2. LOWER BOUNDS

We denote by $\|\cdot\|$, $\|\cdot\|_F$, and $\mathbb{R}$ the spectral norm, the Frobenius norm and the set of real numbers respectively. I is the identity matrix. The following characterization of the spread of a Hermitian matrix is the basis of our analyses.

LEMMA 1. *If A is a Hermitian matrix, then*

$$s(A) = 2\min_{c\in\mathbb{R}}\|A-cI\|. \tag{4}$$

Proof. Let $\lambda_1 \geqslant \cdots \geqslant \lambda_n$ be the eigenvalues of A and $A = U^*DU$ be the spectral decomposition, where U is unitary and $D = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$. Then

$$\begin{aligned} s(A) &= \lambda_1 - \lambda_n = 2 \min_{c \in \mathbb{R}} \max_i |\lambda_i - c| \\ &= 2 \min_c \|D - cI\| \\ &= 2 \min_c \|U^*(D - cI)U\| \\ &= 2 \min_c \|A - cI\|. \end{aligned}$$

The minimum is attained at $c_0 = (\lambda_1 + \lambda_n)/2$. ∎

The next result is well known. See e.g. [6, p. 80].

LEMMA 2. *Let $\|\cdot\|$ be a unitarily invariant norm, and A_0 be any submatrix of an arbitrary matrix A. Then*

$$\|A_0\| \leqslant \|A\|.$$

We just use a special case of this lemma, i.e., the spectral norm. Denote by Im a the imaginary part of a complex number a.

THEOREM 3. *For any Hermitian matrix $A = (a_{ij})$,*

$$s^2(A) \geqslant \max_{i \neq j} \left\{ (a_{ii} - a_{jj})^2 + 2 \sum_{k \neq i} |a_{ik}|^2 + 2 \sum_{k \neq j} |a_{jk}|^2 + 4 e_{ij} \right\}, \tag{5}$$

where

$$e_{ij} = \begin{cases} \left| \sum_{k \neq i, j} \bar{a}_{ik} a_{jk} \right| & \textit{if} \quad a_{ij} = 0, \\ \left| \operatorname{Im}\left(a_{ij} \sum_{k \neq i, j} \bar{a}_{ik} a_{jk} \right) \right| \Big/ |a_{ij}| & \textit{otherwise}. \end{cases} \tag{6}$$

Proof. Let A be $n \times n$. For any real number c and indices $i \neq j$, by Lemma 2 we have

$$\|A - cI\| \geqslant \left\| \begin{pmatrix} a_{i1} & a_{i2} & \cdots & a_{ii} - c & \cdots & a_{in} \\ a_{j1} & a_{j2} & \cdots & a_{jj} - c & \cdots & a_{jn} \end{pmatrix} \right\|. \tag{7}$$

If x and y are row vectors, then

$$\begin{pmatrix} x \\ y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}^* = \begin{pmatrix} xx^* & xy^* \\ yx^* & yy^* \end{pmatrix}.$$

A direct computation shows that

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 = \tfrac{1}{2}\left[xx^* + yy^* + \sqrt{(xx^* - yy^*)^2 + 4|xy^*|^2} \right]. \tag{8}$$

Clearly

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \geqslant \tfrac{1}{2}(xx^* + yy^* + 2|xy^*|). \tag{9}$$

Setting $x = (a_{i1}, \ldots, a_{ii} - c, \ldots, a_{in})$, $y = (a_{j1}, \ldots, a_{jj} - c, \ldots, a_{jn})$ in (9), using $\bar{a}_{ji} = a_{ij}$, and simplifying the expression, we obtain

$$\begin{aligned} \left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 &\geqslant \frac{1}{2}\Bigg(\frac{1}{2}(a_{ii} - a_{jj})^2 + \sum_{k \neq i} |a_{ik}|^2 \\ &\qquad + \sum_{k \neq j} |a_{jk}|^2 + \frac{t^2}{2} + 2\bigg| a_{ij}t + \sum_{k \neq i, j} a_{ik}\bar{a}_{jk} \bigg| \Bigg) \qquad (10) \\ &\geqslant \frac{1}{2}\Bigg(\frac{1}{2}(a_{ii} - a_{jj})^2 + \sum_{k \neq i} |a_{ik}|^2 \\ &\qquad + \sum_{k \neq j} |a_{jk}|^2 + 2\bigg| a_{ij}t + \sum_{k \neq i, j} a_{ik}\bar{a}_{jk} \bigg| \Bigg), \qquad (11) \end{aligned}$$

where $t = a_{ii} + a_{jj} - 2c$ is real.

It is easy to see that for complex numbers a and b with $a \neq 0$,

$$\min_{t \in \mathbb{R}} |at + b| = \frac{|\operatorname{Im}(a\bar{b})|}{|a|}.$$

Thus from (11) we have

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \geqslant \frac{1}{2}\left(\frac{1}{2}(a_{ii} - a_{jj})^2 + \sum_{k \neq i} |a_{ik}|^2 + \sum_{k \neq j} |a_{jk}|^2 + 2e_{ij} \right), \quad (12)$$

where e_{ij} is defined by (6). By Lemma 1, combining (7) and (12) yields (5). ■

If the maximum in (1) is attained at some index pair $i \neq j$, then clearly the bound (5) is sharper than the bound (1), and sometimes it is much sharper, as the examples in Section 3 show. Otherwise, the bound (1) reduces to

$$s^2(A) \geqslant 4 \sum_{k \neq i} |a_{ik}|^2, \qquad i = 1, \dots, n,$$

but this is a special case of the subsequent Theorem 6. Since Mirsky's bound is attained [4], so are (1) and (5). If A is real, a sharper lower bound can be derived.

THEOREM 4. *Let $A = (a_{ij})$ be real and symmetric. Then*

$$s^2(A) \geqslant \max_{i \neq j} \left\{ (a_{ii} - a_{jj})^2 + 2\sum_{k \neq i} a_{ik}^2 + 2\sum_{k \neq j} a_{jk}^2 + 4e_{ij} \right\}, \quad (13)$$

where

$$e_{ij} = \begin{cases} \left| \sum_{k \neq i,j} a_{ik} a_{jk} \right| & \text{if } a_{ij} = 0 \\ \min\left(a_{ij}^2 + \left| 2a_{ij}^2 - \left| \sum_{k \neq i,j} a_{ik} a_{jk} \right| \right|, \dfrac{(\sum_{k \neq i,j} a_{ik} a_{jk})^2}{(2a_{ij})^2} \right) & \text{otherwise}. \end{cases}$$

(14)

Proof. We begin with the inequality (10). Define

$$\phi(t) = \frac{t^2}{2} + 2|at + b|, \qquad a \neq 0, \quad a, b \text{ real}.$$

Then

$$\min_t \phi(t) = \min\left(2a^2 + 2\big|2a^2 - |b|\big|, \frac{b^2}{2a^2}\right). \tag{15}$$

Set $a = a_{ij}$, $b = \sum_{k \neq i, j} a_{ik} a_{jk}$ in $\phi(t)$. Again by Lemma 1, from (7), (10), and (15) we get (13). ∎

Now we make another improvement on (1) by using a similar idea.

THEOREM 5. *Let* $A = (a_{ij})$ *be Hermitian. Then*

$$s^2(A) \geqslant \max_{i \neq j}\left\{(a_{ii} - a_{jj})^2 + 2\sum_{k \neq i} |a_{ik}|^2 + 2\sum_{k \neq j} |a_{jk}|^2 + e_{ij}\right\}, \tag{16}$$

where

$$e_{ij} = \begin{cases} 2f_{ij} & \text{if } a_{ii} = a_{jj} \\ \min\left\{(a_{ii} - a_{jj})^2 + 2\left|(a_{ii} - a_{jj})^2 - f_{ij}\right|, \dfrac{f_{ij}^2}{(a_{ii} - a_{jj})^2}\right\} & \text{otherwise} \end{cases} \tag{17}$$

and $f_{ij} = |\sum_{k \neq i} |a_{ik}|^2 - \sum_{k \neq j} |a_{jk}|^2|$.

Proof. From (8) we have

$$\left\|\begin{pmatrix} x \\ y \end{pmatrix}\right\|^2 \geqslant \tfrac{1}{2}(xx^* + yy^* + |xx^* - yy^*|).$$

Let $x = (a_{i1}, \ldots, a_{ii} - c, \ldots, a_{in})$, $y = (a_{j1}, \ldots, a_{jj} - c, \ldots, a_{jn})$. Then

$$\left\| \begin{pmatrix} x \\ y \end{pmatrix} \right\|^2 \geqslant \frac{1}{2}\left(\frac{1}{2}(a_{ii} - a_{jj})^2 + \sum_{k \neq i} |a_{ik}|^2 + \sum_{k \neq j} |a_{jk}|^2 + \frac{t^2}{2} \right.$$
$$\left. + \left| (a_{ii} - a_{jj})t + \sum_{k \neq i} |a_{ik}|^2 - \sum_{k \neq j} |a_{jk}|^2 \right| \right), \tag{18}$$

where $t = a_{ii} + a_{jj} - 2c$ is real. Define

$$\psi(t) = \frac{t^2}{2} + |at + b|, \qquad a \neq 0, \quad a, b \text{ real}.$$

Then

$$\min_t \psi(t) = \min\left(\frac{a^2}{2} + \left| a^2 - |b| \right|, \frac{b^2}{2a^2} \right). \tag{19}$$

Set $a = a_{ii} - a_{jj}$, $b = \sum_{k \neq i} |a_{ik}|^2 - \sum_{k \neq j} |a_{jk}|^2$. Applying Lemma 1 and combining (7), (18), and (19), we obtain (16). ■

Next we derive a lower bound without involving the diagonal entries. Note that $s(A) = s(A - cI)$ for any real c. Roughly speaking, the spread of a Hermitian matrix is irrelevant to the magnitude of the diagonal entries, but relevant to the differences between them, which cannot be reflected in the next theorem. The bound (2) is of the same flavor.

THEOREM 6. *Let $A = (a_{ij})$ be an $n \times n$ Hermitian matrix. Then*

$$s(A) \geqslant \max_\alpha \frac{2}{\sqrt{|\alpha|}} \sqrt{\sum_{\substack{i \in \alpha \\ j \notin \alpha}} |a_{ij}|^2} \tag{20}$$

where $\varnothing \neq \alpha \subset \{1, 2, \ldots, n\}$ and $|\alpha| \leqslant n/2$.

Proof. It is well known that for any $s \times t$ matrix B,

$$\|B\| \geqslant \frac{1}{\sqrt{\min(s, t)}} \|B\|_F.$$

This relation can be easily seen from the singular values. Denote by $A[\alpha|\beta]$ the submatrix of A consisting of the entries lying in rows $i \in \alpha$ and columns $j \in \beta$. Obviously, if $\alpha \cap \beta = \varnothing$, then $(A - cI)[\alpha|\beta] = A[\alpha|\beta]$ for any c. By Lemma 2 we have

$$\begin{aligned}\|A - cI\| &\geqslant \|(A - cI)[\alpha|\beta]\| = \|A[\alpha|\beta]\| \\ &\geqslant \frac{1}{\sqrt{\min(|\alpha|, |\beta|)}} \|A[\alpha|\beta]\|_F \\ &= \frac{1}{\sqrt{\min(|\alpha|, |\beta|)}} \sqrt{\sum_{\substack{i\in\alpha \\ j\in\beta}} |a_{ij}|^2}.\end{aligned}$$

Setting $\beta = \{1, 2, \dots, n\} \setminus \alpha$, restricting $|\alpha| \leqslant n/2$, and applying Lemma 1, we get (20). ■

In particular, choosing $\alpha = \{i\}$ in Theorem 6, we have

$$s(A) \geqslant 2\sqrt{\sum_{k\neq i} |a_{ik}|^2}.$$

This is the case $i = j$ in the bound (1).

For an $n \times n$ Hermitian matrix A, the following bound is proved in [1, Theorem 3.2].

$$s(A) \geqslant \frac{2}{\sqrt{n}} \sqrt{\|A\|_F^2 - \frac{1}{n}(\operatorname{tr} A)^2} \tag{21}$$

we may give a slightly simpler proof using Lemma 1 and the relation between the spectral norm and the Frobenius norm:

$$\begin{aligned}s(A) &= 2\min_c \|A - cI\| \geqslant \frac{2}{\sqrt{n}} \min_c \|A - cI\|_F \\ &= \frac{2}{\sqrt{n}} \sqrt{\|A\|_F^2 - \frac{1}{n}(\operatorname{tr} A)^2}.\end{aligned}$$

3. EXAMPLES FOR COMPARISONS

We denote the bounds in (1), (2), (5), (13), (20), and (21) by b_1, b_2, b_5, b_{13}, b_{20}, and b_{21} respectively.

3.1. *The Bounds (1) and (5)*

Let

$$A = \begin{pmatrix} 2 & 0 & 1-2i & 1-2i \\ 0 & 1 & 1+2i & 1+2i \\ 1+2i & 1-2i & 1 & 0 \\ 1+2i & 1-2i & 0 & 1 \end{pmatrix}.$$

For this A, $b_1 = 41$ and $b_5 = 81$. The maximum in (1) is attained at $(i, j) = (1, 2)$, $(1, 3)$, and $(1, 4)$ [$i = j$ is permitted in (1)], while the maximum in (5) is attained at $(i, j) = (1, 2)$. It is interesting to note that $b_{21}^2 = 41 - \frac{1}{4}$.

Consider the following matrix which has no zero entries:

$$B = \begin{pmatrix} 5 & 1 & 2-i \\ 1 & 1 & 1+2i \\ 2+i & 1-2i & 3 \end{pmatrix}.$$

For B, $b_1 = 40$ and $b_5 = 60$. The maximum in (1) is attained at $(i, j) = (1, 2)$ or $(3, 3)$. The maximum in (5) is attained at $(i, j) = (1, 2)$. We have $b_{21}^2 = 40$, the same value of b_1.

The above two examples suggest that b_1 and b_{21}^2 are approximately equal for some Hermitian matrices. It is not yet known whether this is always the case.

3.2. *The Bounds (1) and (13) for Real Symmetric Matrices*

Let

$$A = \begin{pmatrix} 2 & 0 & 4 & 3 \\ 0 & 1 & 3 & 4 \\ 4 & 3 & 1 & 0 \\ 3 & 4 & 0 & 1 \end{pmatrix}.$$

For this A, $b_1 = 101$ and $b_{13} = 197$. The maximum in (1) is attained at $(i, j) = (1, 2)$, $(1, 3)$, and $(1, 4)$. The maximum in (13) is attained at $(i, j) = (1, 2)$. We have $b_{21}^2 = 101 - \frac{1}{4}$.

Replacing the zero entries in the above A by $\frac{1}{2}$, we get a nonzero-entry matrix

$$B = \begin{pmatrix} 2 & \frac{1}{2} & 4 & 3 \\ \frac{1}{2} & 1 & 3 & 4 \\ 4 & 3 & 1 & \frac{1}{2} \\ 3 & 4 & \frac{1}{2} & 1 \end{pmatrix}.$$

For this B, $b_1 = 102$ and $b_{13} = 197$. The maximum in (1) is attained at $(i,j) = (1,2)$, $(1,3)$, and $(1,4)$, while the maximum in (13) is attained at $(i,j) = (1,2)$. We have $b_{21}^2 = 102 - \frac{1}{4}$.

3.3. *The Bounds (2) and (20)*

Let

$$A = \begin{pmatrix} * & 3 & 4 & -5 \\ 3 & * & -5 & 1 \\ 4 & -5 & * & 0 \\ -5 & 1 & 0 & * \end{pmatrix}.$$

For this A, $b_2 = 10$ and $b_{20} = 10\sqrt{2}$. One of the maximizing points in (2) is $\alpha = \{1\}$ and $\beta = \{4\}$, while the maximizing point in (20) is $\alpha = \{1\}$.

We remark that sometimes b_2 is sharper than b_{20}, as the following example shows:

$$B = \begin{pmatrix} * & 1 & 1 & 1 \\ 1 & * & 1 & 1 \\ 1 & 1 & * & 1 \\ 1 & 1 & 1 & * \end{pmatrix}.$$

Now, $b_2 = 4$ and $b_{20} = 2\sqrt{3}$. The maximum in (2) is attained at, say, $\alpha = \{1,2\}$ and $\beta = \{3,4\}$, while the maximum in (20) is attained at $\alpha = \{1\}$.

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